Non-invasive technique for the modal analysis of inhomogeneous structures: digital holography and extended Karhunen-Loève Decomposition

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ABSTRACT

The aim of this paper is to present recent applications of a newly developed methodology for the modal analysis of inhomogeneous structures. This is based on the statistical processing of the data extracted from holographic shots of the vibrating object. Specifically, the elastic displacement field is obtained through digital processing of two series of holographic shots (generated by laser beams in quadrature), and an extended Karhunen-Loève Decomposition (KLD) technique is used to obtain base functions that coincide, for unforced vibrations and under certain conditions, with the natural modes of the structure. The extension of the KLD under consideration consists in a different definition of the Hilbert space that embeds the formulation, as one in which the inner product has a weight equal to the density. The coupling of digital holography with this extension of the KLD represents the main novelty of the present work and yields an experimental methodology characterized by several interesting features. First, the use of holographic images as data source provides a non-invasive technique that allows for an accurate analysis of certain phenomena (such as aeroelastic and acoustoelastic problems) for which the instrumentation represents a critical issue. Moreover, the statistical nature of the method makes the results of the analysis independent of the excitation used to initiate the motion. In the present work, the optical holographic process is simulated through a dedicated, in-house developed, computer program and the displacement field is evaluated by finite elements. The KLD technique under consideration may be applied to the analysis of n-dimensional structures ($n = 1, 2, 3$) and is here used to analyze the motion of a two-dimensional structure (cantilever plate). Numerical results reveal that the base functions obtained with the numerical simulation coincide, within plotting accuracy, with the eigenmodes of the structure. Moreover, a numerical technique for the evaluation of the natural frequencies is also shown.

1 INTRODUCTION

The aim of this paper is to present recent applications of a newly developed method for the modal analysis of two-dimensional inhomogeneous structures. This is based on the coupling between digital holographic interferometry and Karhunen-Loève theory, properly extended to the analysis of inhomogeneous structures. Digital holography and Karhunen-Loève decomposition (KLD) have been successfully applied in the past to the modal analysis of homogeneous two-dimesional structures (see Iemma *et al.* [1]). More recently, the standard Karhunen-Loève theory has been extended by Iemma *et al.* [2] to the analysis of inhomogeneous structures. The coupling of digital holography with this extension of the KLD represents the main novelty of the present work. Specifically, the technique presented in Iemma *et al.* [1] is used to obtain the displacement vector of the vibrating structure through digital processing of two series of holographic shots generated by laser beams in quadrature. Then, the displacement vector is processed using the Karhunen-Loève decomposition extended to the analysis of inhomogeneous structures [2] and the approximated natural modes of vibration are evaluated as the eigenfunctions of the so-called Karhunen-Loève integral operator. A numerical technique based on the projection of the displacement vector on the approximated modes is used to calculate the natural frequencies.[1]

Holographic interferometry is an extension of interferometric measurement techniques and it may be used to generate a spatial image of objects deflection. Holographic methods have been successfully applied by Erf [3] in non-destructive testing since 1974 and they are now used wherever deformations and changes in shape of objects have to be evaluated with interferometric accuracy. For the objective of vibrational analysis of elastic structures, the use of holographic images as a data source has two major advantages: *i*) the optical data acquisition is a non-invasive technique which drastically reduce the errors due to the instrumentation; *ii*) the data acquisition covers the whole object domain so that it is possible to store information for a large number of points. In the present work, the method presented in Iemma *et al.* [1] is used to measure the out-of-plane displacements of two-dimensional inhomogeneous structures. This technique may be used to observe arbitrary time-dependent phenomena and is here applied to study a multi-frequency motion using high resolution digital devices.

The Karhunen-Loève Decomposition is a statistical method for finding a base that cover the optimal distribution of energy in the dynamics of a continuum. This method initially appeared in the signal processing literature, where it was presented by Hotelling [4] in 1933 as the Principal Component Analysis (PCA). The theory behind the method was taken again and studied in depth by Kosambi [5] in 1943, by Loève [6] in 1945 and by Karhunen [7] in 1946. Since it was applied by Lumley [8] in 1967 to uncover coherent structures in turbulent flows, it has become a standard tool in turbulence studies [9], where it is also known as the Proper Orthogonal Decomposition (POD). The theory proposed by Karhunen [7] and Loève [6] is recently emerging as a powerful tool in structural dynamics and vibration. A physical interpretation of the use of the KLD in vibrations studies has been shown by Feeny *et al.* [10] and Wolter *et al.* [11]. In structural dynamics, the method consists in constructing the time-averaged spatial autocorrelation tensor of the elastic displacement field of the structure. Its spectral analysis produces a basis, as a set of orthonormal eigenfunctions (eigenvectors, in the numerical approach) and the corresponding set of eigenvalues, which represent the energy content of each mode. It has been shown (*e.g.*, Ref. [10]) that for undamped and unforced structures with constant density, the eigenfunctions given by the standard KLD coincide with the natural modes of vibration. Recently, the formulation has been extended by Iemma *et al.* [2] to the modal identification of structures with non-uniform density. It is worth noting that this extension of the KLD may be applied to the modal analysis of n-dimensional structures ($n = 1, 2, 3$). In this work, the technique presented in Iemma *et al.* [2] is coupled with a measurement technique able to analyze the motion of two-dimensional structures (see later) and is applied to the modal identification of a cantilever plate ($n = 2$).

In the next sections, the interferometric technique used to evaluate the displacement vector is briefly outlined, then the general theory underlying the Karhunen-Loève decomposition is recalled, with emphasis on its application to quasi-periodic dynamical systems with non-uniform density. The assumptions made in structural dynamics are shown and the extented KLD is outlined. A method for estimating the natural frequencies is also shown and the numerical results are presented.

2 DISPLACEMENT VECTOR FROM OPTICAL FIELD

In this section, we briefly outline the basic optics analytical tools relevant to the understanding of the present methodology. In this application, we consider a thin deformable plate, lying in the xy plane, capable of out-of-plane deformation. When a laser beam hits the object at rest, the reflected optical field may be considered as the effects of a sources distribution $V_r(x, y, z_o)$ on the object plane $z = z_o$. On the other hand, indicating with $w(x, y)$ the out-of-plane displacement due to a (static) deformation, the complex representation of the relationship between the distribution $V_r(x, y, z_o)$ and that emanating from the deformed plate is

$$V(x, y, z_o) = V_r(x, y, z_o)\, e^{\imath k_z w(x,y)}, \tag{1}$$

[1] Or damped frequencies, if damped vibrations occur.

where k_z is the wave number of the coherent light in the z direction. For a vibrating object, Eq. (1) becomes

$$V(x, y, z_o, t) = V_r(x, y, z_o)\, e^{\imath k_z w(x,y,t)}. \tag{2}$$

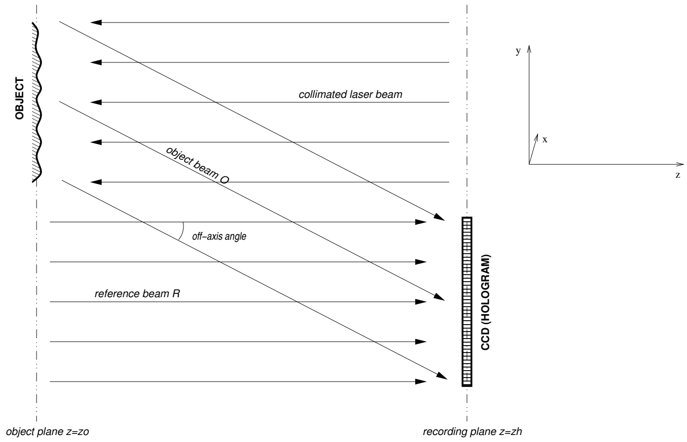

Figure 1: Scheme of the holographic optical setup

The complete information about the two fields $V_r(x, y)$ and $V(x, y)$ may be recorded on a suitable device, such as a photographic film or a digital CCD, using a real-time holographic technique (for details, see Refs. [12, 13]). To this aim, the recording device is illuminated by two different laser beams coming, respectively, from the object (object beam $O(x, y, z, t)$ produced by the sources given by Eq. 1 or Eq. 2, see Fig. 1), and from a known source (reference beam $R(x, y, z)$, see again Fig. 1). The resulting optical field intensity (*i.e.*, the quantity the recording medium is sensible to) may be expressed by the superposition, on the holographic plane $z = z_h$, of the two incoming beams:

$$\begin{aligned} I_h(x, y, t) &= |R(x, y, z_h) + O(x, y, z_h, t)|^2 \\ &= |R(x, y, z_h)|^2 + |O(x, y, z_h, t)|^2 \\ &+ R(x, y, z_h)\, O^*(x, y, z_h, t) + R^*(x, y, z_h)\, O(x, y, z_h, t), \end{aligned} \tag{3}$$

where $z = z_h$ is the recording plane and * indicates the complex conjugate. To reconstruct completely the displacement field from the recorded data, we need to know the complex optical field. To accomplish this, it is convenient to record two different simultaneous holograms, using two reference beams in quadrature

$$R_0(x, y, z_h) = A, \qquad R_{\frac{\pi}{2}}(x, y, z_h) = A\, e^{\imath \frac{\pi}{2}}, \tag{4}$$

where A is a real constant. The corresponding intensities on the holographic planes are respectively

$$I_{h0}(x, y, t) = A^2 + |O(x, y, z_h, t)|^2 + 2A\, \mathrm{Re}\big[O(x, y, z_h, t)\big], \tag{5}$$

$$\begin{aligned} I_{h\frac{\pi}{2}}(x, y, t) &= A^2 + |O(x, y, z_h, t)|^2 + 2A\, \mathrm{Re}\big[O(x, y, z_h, t)e^{-\imath\pi/2}\big] \\ &= A^2 + |O(x, y, z_h, t)|^2 + 2A\, \mathrm{Im}[O(x, y, z_h, t)]. \end{aligned} \tag{6}$$

The recording of the optical filed intensity on the sensitive medium implies a time-integration over the exposure time T_E of Eqs. 5 and 6 (see *e.g.*, Ref.[12]). In this work, T_E is assumed much smaller than the period of oscillation of the highest harmonics of the motion analyzed,[2] and the medium response is assumed proportional to the identity operator. Thus, the fields recorded are considered proportional to the optical field intensities in Eqs. 5 and 6. Then, since A is known and $|O(x, y, z_h, t)|^2$ can be easily measured, $\mathrm{Re}\big[O(x, y, z_h)\big]$ and $\mathrm{Im}\big[O(x, y, z_h, t)\big]$ may be evaluated using the above equations.

[2]The possibility to satisfy such a condition in practical applications depends on a large number of parameters related to the physical phenomenon and to the characteristics of the experimental rig (*e.g.*, medium sensitivity, laser power, object dimensions, etc.).

2.1 Digital reconstruction and displacement field estimate

As shown above, we may obtain the complex optical fields $O_r(x, y, z_h)$ and $O(x, y, z_h, t)$ coming respectively from the object at rest and from the vibrating object. Then we may reconstruct the original fields on the object plane via digital simulation of laser light propagation. According to Huygens-Fresnel principle we may write

$$V_r(x, y, z_o) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K^*(x-\xi, y-\eta, z_o - z_h)\, O_r(\xi, \eta, z_h)\, d\xi d\eta, \tag{7}$$

$$V(x, y, z_o, t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K^*(x-\xi, y-\eta, z_o - z_h)\, O(\xi, \eta, z_h, t)\, d\xi d\eta, \tag{8}$$

where

$$K(x, y, z) = -\frac{\imath e^{ikr}}{\lambda r}\, \cos\vartheta, \tag{9}$$

where λ is wavelength of the laser light, $r = \|\mathbf{r}\|$ (with $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$) and ϑ is the angle between $\mathbf{r}$ and the z-axis. Once $V_r(x, y, z_o)$ and $V(x, y, z_o, t)$ are known, $w(x, y, t)$ may be evaluated through Eq. (2).

3 EXTENDED KARHUNEN-LOEVE DECOMPOSITION

In structural dynamics, the method introduced by Karhunen and Loève is used to provide a basis for the *optimal* representation of the displacement vector $\mathbf{u}(\mathbf{x}, t)$ of a vibrating inhomogeneous structure. The method provides a basis which is optimal, in the energy content sense, for the representation of the displacement vector $\mathbf{u}(\mathbf{x}, t)$ in the linear combination $\mathbf{u}(\mathbf{x}, t) = \sum_{k=1}^{n} \beta_k(t)\, \boldsymbol{\varphi}_k(\mathbf{x})$, truncated to the order n, with $\mathbf{x} \in \mathcal{D}$ and $t \in [0, T]$.[3] The optimality condition associated to the KLD ensures that, for a given n, the first n KLD basis functions capture, on average, more energy than any other orthonormal basis in the linear representation of the field $\mathbf{u}$ (see, *e.g.*, Holmes *et al.* [9]). It has been shown that this property is satisfied (under certain conditions) by the natural modes, provided that the formulation is embedded in the proper Hilbert space (see Iemma *et al.* [2]). In the following, the theory underlying the extension of the KLD to the modal identification of inhomogeneous structures is briefly recalled.

We assume that the dynamics of the undamped-unforced system is governed by the equation $\rho(\mathbf{x})\, \ddot{\mathbf{u}}(\mathbf{x}, t) + \mathcal{L}\, \mathbf{u}(\mathbf{x}, t) = 0$, where $\rho = \rho(\mathbf{x})$ is the structure density. Thus, the displacement vector is given by $\mathbf{u}(\mathbf{x}, t) = \sum_{k=1}^{\infty} \alpha_k(t)\, \boldsymbol{\phi}_k(\mathbf{x})$, where $\boldsymbol{\phi}_k(\mathbf{x})$ are the natural modes of vibration (linear normal modes), solution of $\mathcal{L}\, \boldsymbol{\phi}_k(\mathbf{x}) = \rho(\mathbf{x})\, \mu_k\, \boldsymbol{\phi}_k(\mathbf{x})$, with $\int_{\mathcal{D}} \rho(\mathbf{x})\, \boldsymbol{\phi}_i(\mathbf{x}) \cdot \boldsymbol{\phi}_j(\mathbf{x})\, \mathrm{d}\mathbf{x} = \delta_{ij}$. The time dependency of the solution is given by $\alpha_k(t) = a_k\, \cos(\omega_k t + \chi_k)$, where $\omega_k = \sqrt{\mu_k}$, and a_k, $\chi_k \in \Re$ are determined by the initial conditions.

Assuming that the displacement vector (at a given time) belongs to the Hilbert space $L^2_\rho(\mathcal{D})$, defined by the inner product $(\mathbf{f}, \mathbf{g})_\rho := \int_{\mathcal{D}} \rho(\mathbf{x})\, \mathbf{f}(\mathbf{x}) \cdot \mathbf{g}(\mathbf{x})\, \mathrm{d}\mathbf{x}$, the optimal decomposition of the vector $\mathbf{u}$ is given by the solutions of the integral problem (a complete proof of the following equation is given in Ref. [2] and, thus, not repeated here)

$$\mathcal{L}^E_{\mathbf{R}}\, \boldsymbol{\varphi}(\mathbf{x}) := \int_{\mathcal{D}} \rho(\mathbf{y})\, \mathbf{R}(\mathbf{x}, \mathbf{y})\, \boldsymbol{\varphi}(\mathbf{y})\, d\mathbf{y} = \lambda\, \boldsymbol{\varphi}(\mathbf{x}). \tag{10}$$

where $\mathbf{R}(\mathbf{x}, \mathbf{y}) := \langle \mathbf{u}(\mathbf{x}, t) \otimes \mathbf{u}(\mathbf{y}, t) \rangle$ is the time-averaged autocorrelation tensor of the displacement vector $\mathbf{u}(\mathbf{x}, t)$, being $\langle \ldots \rangle := \int_0^T \ldots \mathrm{d}t$ the time-averaging operator and $\otimes$ the standard tensor product. $\mathcal{L}^E_{\mathbf{R}}$ is the extended Karhunen-Loève integral operator and the KLD optimal basis is given by its eigensolutions. It may be shown that $\mathcal{L}^E_{\mathbf{R}}$ is selfadjoint in $L^2_\rho(\mathcal{D})$, *i.e.*, $(\mathbf{f}, \mathcal{L}^E_{\mathbf{R}}\, \mathbf{g})_\rho = (\mathcal{L}^E_{\mathbf{R}}\, \mathbf{f}, \mathbf{g})_\rho$, and compact (since the kernel of Equation 10 is bounded). Hence, its eigenvalues are real and its eigenfunctions form a complete set of orthogonal functions in the above-defined Hilbert space (see *e.g.*, Ref. [14]). Under the hypothesis of undergoing unforced free vibrations

[3] Note that, in general, $\mathbf{x} \in \mathbb{E}^n, n = 1, 2, 3$ and $\mathbf{u}(\mathbf{x}, t) \in \mathbb{V}^m, m = 1, 2, 3$, being $\mathbb{E}^n$ an n-dimensional ($n = 1, 2, 3$) Euclidean point space and $\mathbb{V}^m$ an m-dimensional ($m = 1, 2, 3$) vector space, with n not necessarily equal to m; consider, for instance, the case of a bending beam ($n = 1, m = 2$), or of a bending plate ($n = 2, m = 1$).

and assuming an observation time T tending to infinity, the Karhunen-Loève eigenfunctions coincide with the natural modes of the structure, *i.e.*, $\boldsymbol{\varphi}_k(\mathbf{x}) = \boldsymbol{\phi}_k(\mathbf{x})$ and, in addition, $\lambda_k = \frac{1}{2}\, a_k^2$ (again, see Iemma *et al.* [2]). In practical applications, proper modal identification has to be expected if the observed vibration is representative of the motion from a statistical point of view. This is ensured if all the modes present in the motion undergo a sufficient number of periods during the acquisition time. Thus, the acquisition time has to be sufficiently long provided, of course, that damping is not too high (see also Ref. [15]).

4 NATURAL FREQUENCIES ESTIMATE

The natural frequencies are evaluated using the following technique. First, the coefficients $\beta_k(t)$ (see previous Section) are computed as the $L^2_\rho(\mathcal{D})$-projections of the vector $\mathbf{u}(\mathbf{x}, t)$ onto the k-th Karhunen-Loève mode, *i.e.*, $\beta_k(t) = (\mathbf{u}, \boldsymbol{\varphi}_k)_\rho := \int_{\mathcal{D}} \mathbf{u}(\mathbf{x}, t) \cdot \boldsymbol{\varphi}(\mathbf{x}) d\mathbf{x}$. Then the above coefficients are Fourier-transformed. Under the hypothesis of proper modal identification (*i.e.*, under the hypothesis that $\boldsymbol{\phi}_k = \boldsymbol{\varphi}_k$) the frequency associated to the k-th Karhunen-Loève mode is evaluated and assumed as the natural frequency associated to the corresponding natural mode[4] (see *e.g.*, Refs. [11] and [1]).

5 NUMERICAL RESULTS

In this section, numerical results obtained with a digital simulation of the whole analysis process are presented. The motion, evaluated by finite elements, is used as an input. The holographic recording and reconstruction are simulated using an in-house developed computer program. Specifically, we assume that the object dimensions are small with respect to the the distance $d = |z_h - z_o|$ (see Fig. 1), and thus the Fresnel approximation to the optical propagation laws is applicable (for details, see Ref. [13]). Furthermore, the response of the recording media is described by the identity operator. Finally, the displacement vector evaluated using the technique described in Section 2, is used as the input of the extended KLD (EKLD) in order to evaluate the natural modes of vibration. The natural frequencies are computed using the methodology presented in the previous Section.

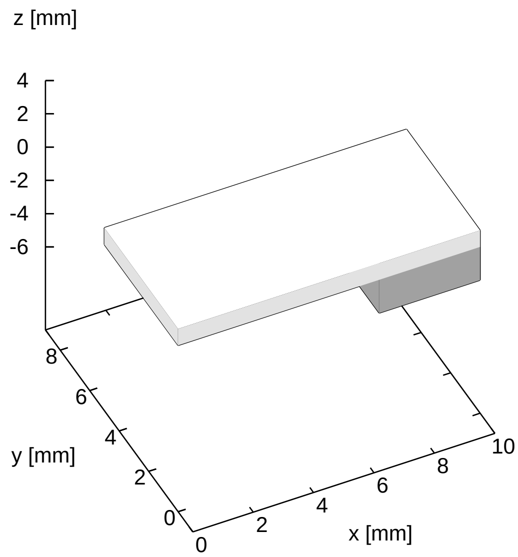

Figure 2: Cantilever plate (clamped on the $x = 0$ section)

[4]If damped vibrations occur, the method gives the so-called damped frequency.

We consider a 10 mm x 5 mm cantilever plate having non uniform thickness (this results in a non-uniform mass per unit area, see Fig. 2). We consider only the bending deformation of the plate, so that the vertical displacement w corresponds to that of a cantilever beam. The solution, truncated to the order M, is given by

$$w(x, y, t) = \sum_{k=1}^{M} a_k \cos(\omega_k t + \theta_k)\, \phi_k(x) \tag{11}$$

where the natural modes $\phi_k(x)$ and the natural frequency ω_k are evaluated using a FEM code, whereas a_k and θ_k are determined by arbitrary initial conditions. The displacement time-history, used here as the input of the simulation process, is calculated using the first ten natural modes ($M = 10$).

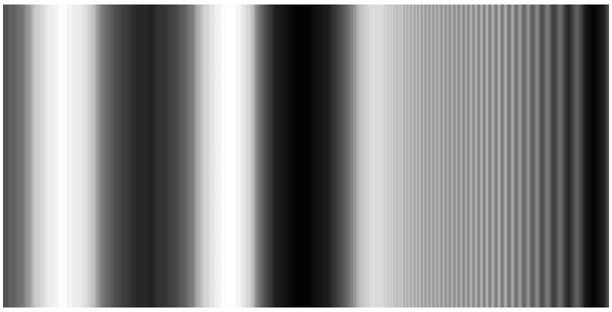

Figure 3: Holographic plane: real part of the optical field

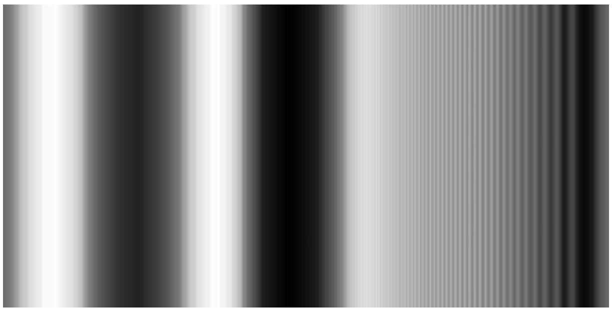

Figure 4: Holographic plane: imaginary part of the optical field

The optical fields are evaluated over a 1024 x 512 9.8μm square pixels array. Figures 3 and 4 show respectively the density plot of the real and the imaginary part of the optical field on the holographic plane, for $t = t_0$. The digital reconstruction is shown in Figs. 5 and 6 where the real part and the imaginary part of the optical field on the object plane are respectively depicted.

The technique discussed in Section 2 is used to evaluate the displacement vector (Fig. 7). Figure 8 shows a comparison between the extracted displacement vector and the displacement vector used as the input, resulting in a very good agreement.

The displacement vector is down-sampled in a 32 x 8 points mesh and processed using the extension of the KLD. Figure 9 shows the first ten eigenvalues given by the method. The remaining eigenvalues are numerically zero (note that this was expected since only ten modes are present in the input). Figures 10, 11, 12, 13 show respectively the third, fift, seventh and tenth mode given by the extended KLD.

Figure 14 depicts a comparison between the fifth mode evaluated using the Karhunen-Loève decomposition and the mode used for the input of the simulation. The results of both standard and extended KLD are presented, showing the good effect on the results of the extended formulation.

Figure 15 presents a comparison between the tenth mode extracted by the EKLD and the tenth input mode. The results of two different experiments are shown. In the first exercise, a sampling frequency of $256,200$ Hz is

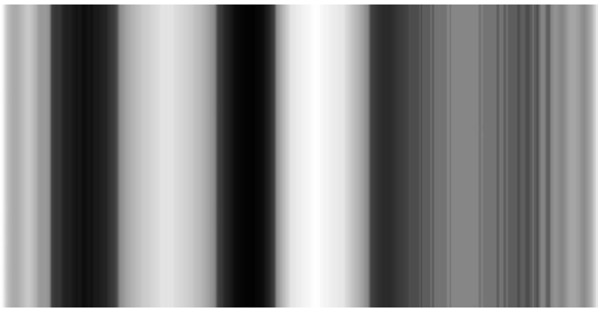

Figure 5: Object plane: real part of the optical field

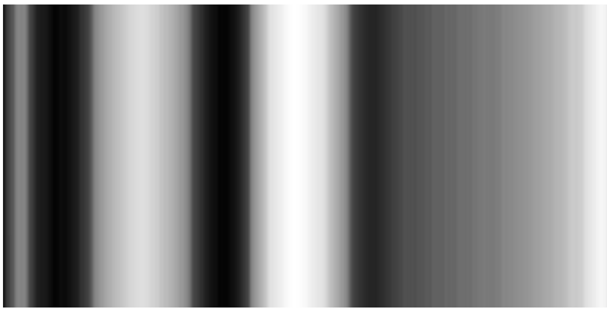

Figure 6: Object plane: imaginary part of the optical field

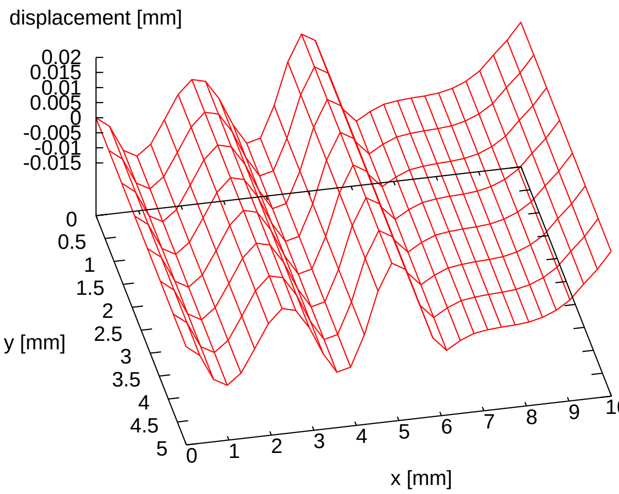

Figure 7: Displacemet vector evaluated form optical field

used to evaluate the time-history of the displacement vector. The sampling frequency used equals the Nyquist frequency for the tenth mode and the results are in a very good agreement with the FEM modes used for the input. In the second experiment, a sampling frequency of 1 Hz is assumed and, thus, Shannon's theorem for time-sampling is not satisfied. The results are in an excellent agreement with the previous ones and, again, in a very good agreement with the input modes. It is worth noting that the KLD acts as an averaging operator

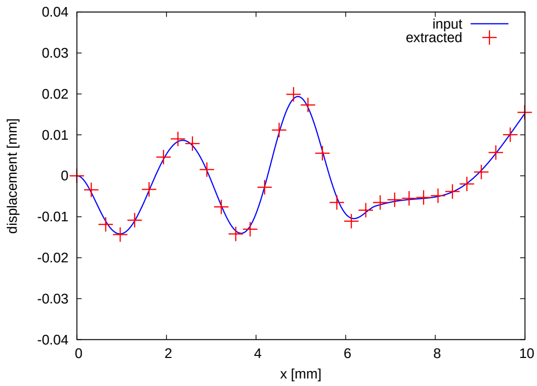

Figure 8: Comparison between extracted and input displacements

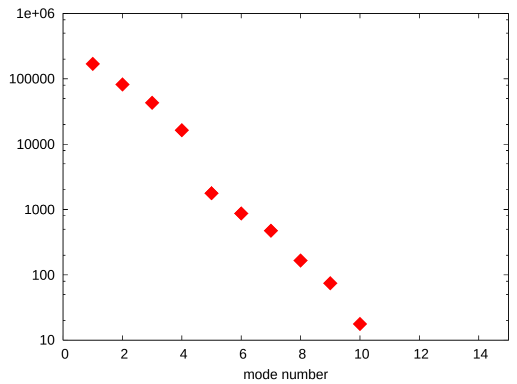

Figure 9: Karhunen-Loève eigenvalues

in the time domain (see Eq. 10). Thus, the time-sampling frequency is not a relevant parameter for modal identification and, as long as the time average of the signal converges to its theoretical value,[5] it is possible to perform the KLD analysis without satisfying Shannon's time-sampling theorem. Obviously this would not be acceptable if, in addition, the natural frequencies were to be required. It may be added that, since the KLD is a statistical method, the number of time samples available (rather than time-sampling frequency), along with the number of spatial samples, are the crucial parameters to achieve a proper modal identification.

The technique presented in Section 4 is used to evaluate the natural frequencies. Accordingly, the displacement vector sampled at 256,200 Hz is processed and its projection on the EKLD fifth mode is shown in Fig. 16. The power spectral density of the projection is shown in Fig. 17. The peak occurs at 28,591Hz whereas the frequency evaluated by finite elements and used for the input is 28,604 Hz. The resulting error is less than 0.05%.

[5]In other words, as long as the time samples used are statistically representative of the motion observed.

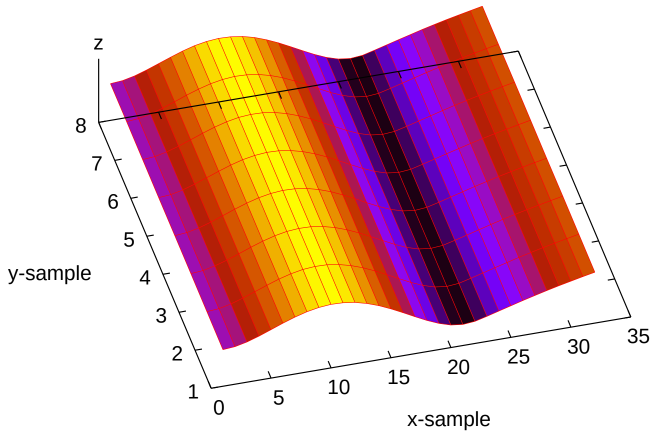

Figure 10: EKLD mode 3

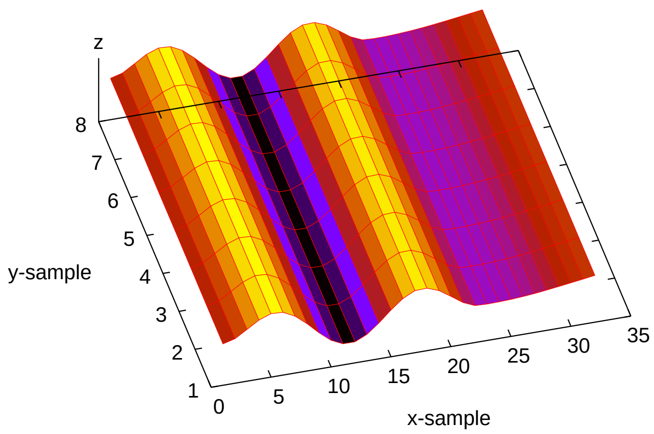

Figure 11: EKLD mode 5

6 CONCLUDING REMARKS

An accurate and non-invasive method based on the coupling between digital holography and extended Karhunen-Loève decomposition (EKLD) for the modal analysis of inhomogeneous structures has been presented. In this work, the whole analysis process has been simulated through an in-house developed computer program. The motion of the structure, used as input of the simulation, has been evaluated by finite elements. Applications to the analysis of a simple two-dimensional structure reveal a good agreement between the empirical eigenfunctions and the input modes. A method for finding the natural frequencies has been used, showing a remarkable agreement between the frequencies evaluated through the present technique and the frequencies used for the

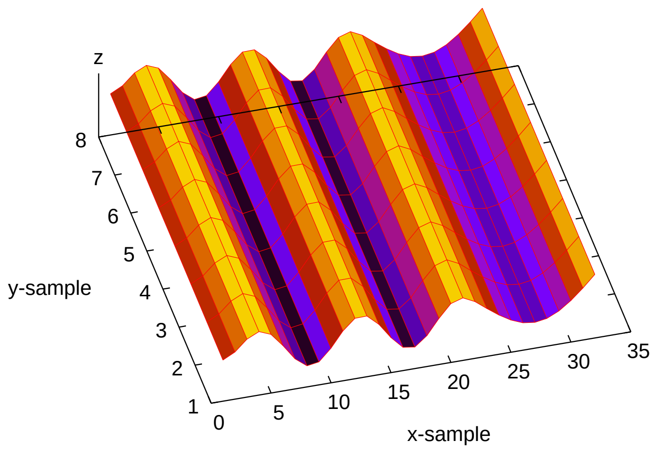

Figure 12: EKLD mode 7

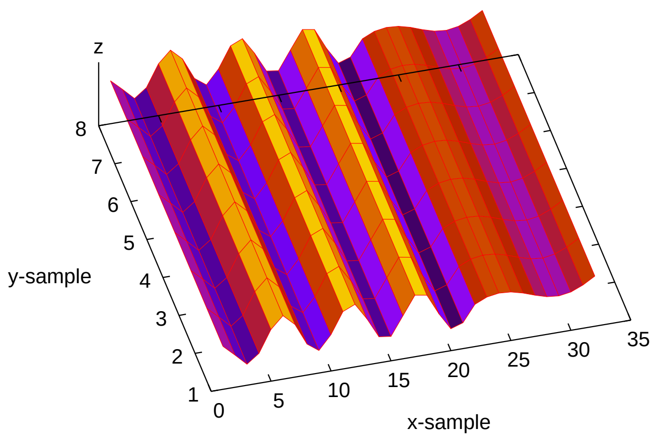

Figure 13: EKLD mode 10

input.

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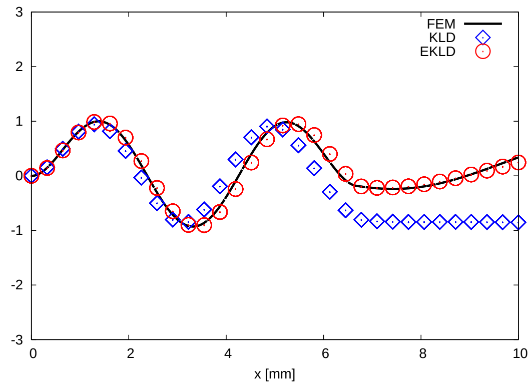

Figure 14: Comparison between fifth KLD mode (standard and extended formulation) and input mode

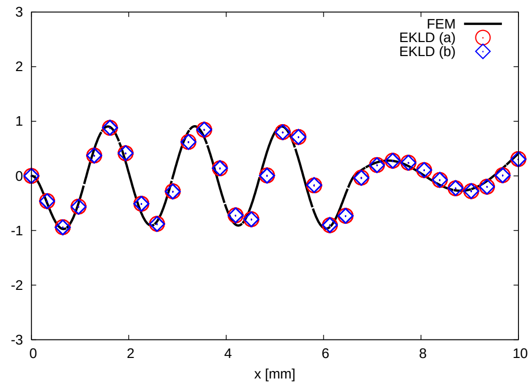

Figure 15: Comparison between tenth EKLD mode and input mode: (a) 256,200 Hz sampling frequency and (b) 1 Hz sampling frequency

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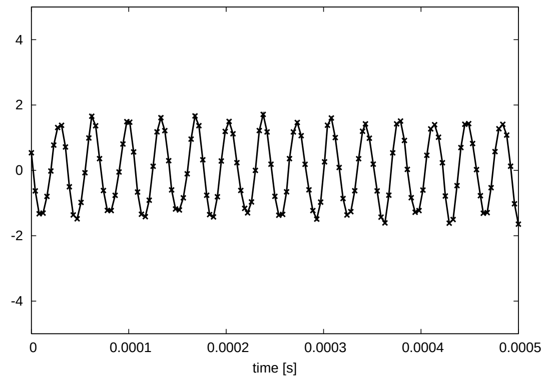

Figure 16: Projection of the displacement vector on EKLD mode 5

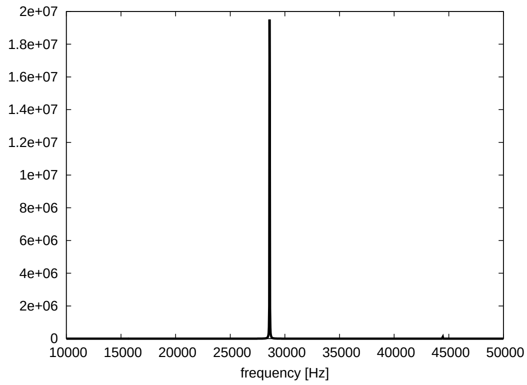

Figure 17: PSD of the projection on the EKLD mode 5

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